**First semestral backpaper exam 2011**
**B.Math. (Hons.) IInd year**
**Algebra III — B.Sury**
**Answer any FIVE questions.**

**Q 1.** Prove that $\mathbf{Z}[i]$ is a PID.
**Q 2.** Let $A$ be a commutative ring with unity and $f \in A[X]$. If $f$ is a zero divisor, prove that there exists $0 \neq a \in A$ such that $af = 0$.
**Q 3.** Let $I_1, \cdots, I_n$ be ideals of a commutative ring $A$ with unity. If $P$ is a prime ideal of $A$ containing the product $I_1 I_2 \cdots I_n$, then show that $P$ contains $I_i$ for some $i$.
**Q 4.** Prove that all ideals of $\mathbf{Z}[X]$ are finitely generated.
**Q 5.** Let $\theta : \mathbf{C}[X, Y] \to \mathbf{C}[T]$ be the ring homomorphism given by $X \mapsto T^2, Y \mapsto T^3$. Prove that Ker $\theta = (X^3 - Y^2)$.
**Q 6.** Let $A = \{a/b \in \mathbf{Q} : b \text{ odd } \}$. Consider $\mathbf{Q}$ as an $A$-module. Show that the unique maximal ideal $m$ of $A$ satisfies $m\mathbf{Q} = \mathbf{Q}$. Why does this not contradict the Nakayama lemma.
**Q 7.** Given a matrix $M \in M_n(K)$, where $K$ is a field, what is meant by its rational canonical form? Further, by assuming the existence of the rational canonical form, compute the characteristic polynomial of $M$.